Quiz 1 Solutions

1. Give counterexamples to disprove the following statements about sets in $\mathbb{R}^n$.

   (a) The intersection of an infinite collection of open sets is open.

   (b) The union of an infinite collection of closed sets is closed.

   Solution. (a) Consider the collection $\{B(0; \frac{1}{n})\}_{n\in\mathbb{N}}$ of open balls in $\mathbb{R}^n$ centered at origin. It is easy to see that

   $$\bigcap_{n\in\mathbb{N}} B(0; \frac{1}{n}) = \{0\},$$

   which is a finite set, and hence a closed set of $\mathbb{R}^n$.

   (b) Consider any open ball $B(a; r)$ in $\mathbb{R}^n$. Please note that any open ball in $\mathbb{R}^n$ will have infintely many points. Moreover, it is clear that

   $$B(a; r) = \bigcup_{x\in B(a;r)} \{x\},$$

   where each $\{x\}$ is a closed set of $\mathbb{R}^n$.

   Alternatively, we can take the collection of closed sets $\{B(0; \frac{1}{n})^c\}_{n\in\mathbb{N}}$. By applying, De-Morgan's Law in (a), we have

   $$\bigcup_{n\in\mathbb{N}} B(0; \frac{1}{n})^c = \mathbb{R}^n \setminus \{0\},$$

   which is an open set in $\mathbb{R}^n$.

2. Let $f(x, y) = 0$, if $y \leq 0$ or if $y \geq x^2$ and let $f(x, y) = 1$ if $0 < y < x^2$.

   (a) Show that $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along any straight line through the origin.

   (b) Determine whether f is continuous at $(0, 0)$.

   Solution. (a) Any straight line through the origin will be of the form $y = mx$.

   Case 1: Let $m = 0$. Then $y = 0$, then $f(x, 0) = 0$, for all x. Therefore, the limit along the line $y = 0$ is 0.

Case 2: Assume that $m > 0$. For sufficiently small $x \geq 0$, we have $mx \geq x^2$, and hence $f(x, mx) = 0$. For $x < 0$, $y < 0$, and therefore $f(x, y) = 0$. So the limit along $y = mx$ will be 0 in this case.

Case 3: Assume that $m < 0$. For $x \geq 0$, we have $mx \leq 0$, and hence $f(x, y) = 0$. For $x < 0$ and sufficiently close to 0, we have $mx > x^2$, and hence $f(x, y) = 0$. Therefore, the limit along $y = mx$ will be 0 in this case.

(b) Consider the curve $y = x^2/2$, for $x > 0$. Then $y < x^2$, and consequently, $f(x, y) = 1$ along the curve. Therefore, the limit is also 1 along the curve. From this and (a) we can conclude that f is not continuous.